11.

Transformation matrices

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Transformations and matrices

Given some point in the cartesian plane with coordinates (x_1, y_1) we could transform this point to some point (x_2, y_2) using a 2×2 matrix, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, as follows:

$$\begin{aligned}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \\ &= \begin{bmatrix} ax_1 + by_1 \\ cx_1 + dy_1 \end{bmatrix}\end{aligned}$$

In this way 2×2 matrices can be used to represent transformations.

Note that in this context we stand the coordinates of the point up as a column matrix and premultiply by the 2×2 **transformation matrix**.

- A transformation, T, is said to be **linear** if $\quad \mathrm{T}\begin{bmatrix} kx_1 \\ ky_1 \end{bmatrix} = k\mathrm{T}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$

  and $\quad \mathrm{T}\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} = \mathrm{T}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \mathrm{T}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}.$

  The reader is left to confirm that this is the case for $\quad \mathrm{T} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$

- In the next example the images of points A, B and C under some transformation are written A′, B′ and C′, i.e. A *dash*, B *dash* and C *dash*. This dash notation commonly used for the image of a point under a transformation should not be confused with the use of a dash to indicate differentiation, a topic you will meet in the *Mathematics Methods* course, where we write $f'(x)$ for the derivative of $f(x)$ with respect to x. Which meaning is to be attributed to the dash will usually be obvious.

EXAMPLE 1

The points A(2, 1), B(3, –2) and C(0, 1) are transformed to A′, B′ and C′ by the matrix $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

Find the coordinates of A′, B′ and C′.

Solution

$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$. Thus A′ has coordinates (4, 1).

$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$. Thus B′ has coordinates (–1, –2).

$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Thus C′ has coordinates (2, 1).

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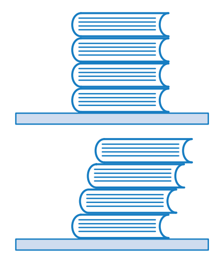
Note: • All 2 × 2 matrices will transform the origin to itself:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The origin is an **invariant** point under these transformations.

It follows that translations cannot be represented by 2 × 2 matrices because, under a translation, the origin is moved. Indeed a translation of 'p units right and q units up' would be achieved by adding the column matrix $\begin{bmatrix} p \\ q \end{bmatrix}$, not by attempting to multiply by a 2 × 2 matrix.

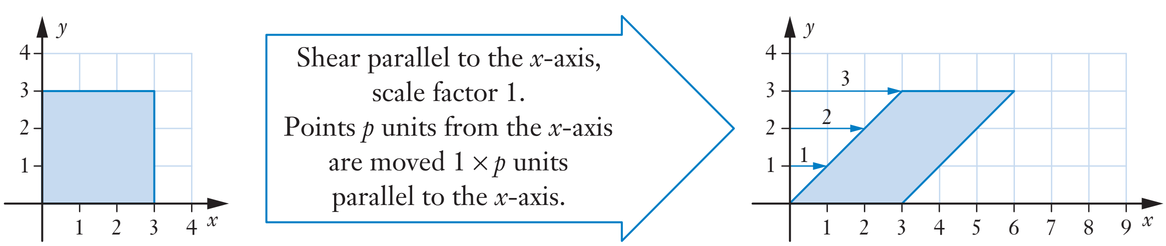
- We will concentrate on matrices that perform reflections in lines passing through the origin, rotations about the origin, dilations and, going a little beyond the confines of the syllabus, shears. (The next dot point explains a *shear*.) In all of these transformations, straight lines are transformed to straight lines. i.e. Points that are collinear before the transformation will still be collinear after the transformation. However, we will also see some matrices that 'flatten' *all* points in the x-y plane onto a line, and some that 'collapse' all points lying on a line onto a single point.
- To understand a **shear**, consider four books stacked on a table as shown on the right. Suppose now that we move the top three books until the stack has the staggered arrangement shown in the lower diagram.

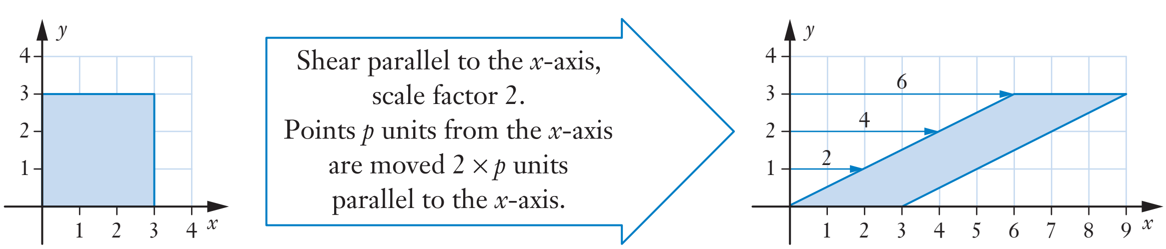


Notice that the higher up the pile a book is, the more it has been moved right. The bottom book remains in its original position and has not been moved at all.

In a shear transformation we have an invariant line which points move parallel to. The further a point is from the invariant line the more that point moves.

If the scale factor of the shear is k then a point that is p units from the invariant line moves a distance kp parallel to the invariant line.





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EXAMPLE 2

By considering the effect on the rectangle O(0, 0), A(2, 0), B(2, 1), C(0, 1), determine the transformation represented by the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Solution

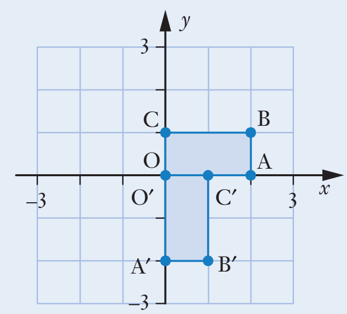
O′: $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

A′: $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$.

B′: $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

C′: $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Plotting OABC and O′A′B′C′ on a graph, see right, we see that the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ represents a clockwise rotation about the origin of 90°. (Or, if we use the convention of anticlockwise rotations being positive, a rotation of –90°, or 270°, about the origin.)



Exercise 11A

By considering the effect on the rectangle O(0, 0), A(2, 0), B(2, 1), C(0, 1) determine the transformation represented by each of the following matrices.

1 $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ **2** $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ **3** $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

4 $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ **5** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **6** $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

7 $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ **8** $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ **9** $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

10 $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ **11** $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ **12** $\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

13 For each of the matrices of questions 1 to 12 determine

a the absolute value of the determinant of the matrix.

b the value of $\dfrac{\text{Area O}'\text{A}'\text{B}'\text{C}'}{\text{Area OABC}}$ where O′, A′, B′ and C′ are the respective images of O, A, B and C under the transformation.

c What do you notice?

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Determining the matrix for a particular transformation

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With a bit of thought, it is easy to write down the matrix corresponding to each transformation you encountered in the last exercise. The method uses the fact that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$$

and $$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}.$$

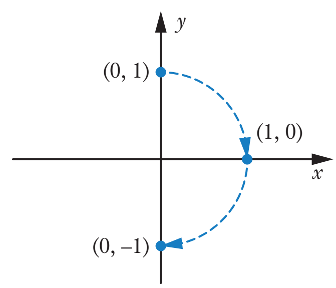
i.e. The image of (1, 0) under the transformation gives us the first column of the matrix and the image of (0, 1) gives us the second column.

For example

- If we required the matrix that represents a 90° clockwise rotation about the origin, we note that under this transformation:

  (1, 0) maps to (0, –1) and (0, 1) maps to (1, 0).

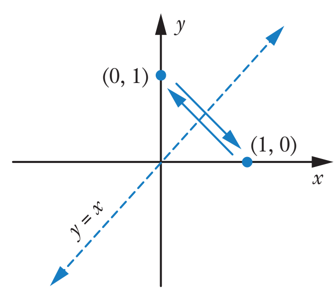
  The required matrix is $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.



- If we required the matrix that represents a reflection in the line $y = x$, we note that under this transformation:

  (1, 0) maps to (0, 1) and (0, 1) maps to (1, 0).

  The required matrix is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.



The determinant of a transformation matrix

Question 13 of Exercise 11A should have led you to conclude that when a shape with area A is transformed by a matrix with determinant k, the area of the image is $|k|$A.

The inverse of a transformation matrix

If matrix T transforms point A to its image A′ then T^{-1}, the multiplicative inverse of T, will transform A′ back to A (provided of course that T^{-1} exists).

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EXAMPLE 3

The triangle ABC is transformed to A′B′C′ by the transformation matrix T where

$$T = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}.$$

If A′, B′ and C′ have coordinates (22, 10), (–3, –5) and (–13, –5) respectively find the coordinates of A, B and C.

If triangle A′B′C′ has an area of 75 units^2 determine the area of triangle ABC.

Solution

$$\left(\text{Note that if } T = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \text{then } T^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}\right)$$

If A has coordinates (a, b)

$$\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 22 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 22 \\ 10 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 12 \\ 8 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

If B has coordinates (c, d)

$$\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} c \\ d \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -3 \\ -5 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 \\ -12 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \end{bmatrix}$$

If C has coordinates (e, f)

$$\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} -13 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} e \\ f \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -13 \\ -5 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -8 \\ -2 \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \end{bmatrix}$$

T has a determinant of $(3)(1) - (1)(1) = 2$

$\therefore$ Area $\triangle A'B'C' = 2$ (Area $\triangle ABC$)

Area $\triangle ABC = 37.5$ units^2

The coordinates of A, B and C are (6, 4), (1, –6) and (–4, –1) respectively and triangle ABC has an area of 37.5 units^2.

Combining transformations

Suppose that (x, y) is transformed to (x', y') by matrix P and (x', y') is transformed to (x'', y'') by matrix Q.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \mathrm{P}\begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{and} \qquad \begin{aligned} \begin{bmatrix} x'' \\ y'' \end{bmatrix} &= \mathrm{Q}\begin{bmatrix} x' \\ y' \end{bmatrix} \\ &= \mathrm{QP}\begin{bmatrix} x \\ y \end{bmatrix} \end{aligned}$$

Thus the single matrix equivalent to applying 'P followed by Q' is QP.

This may initially seem the wrong way around but remember that in the expression $\mathrm{QP}\begin{bmatrix} x \\ y \end{bmatrix}$ it is matrix P that operates on $\begin{bmatrix} x \\ y \end{bmatrix}$.

EXAMPLE 4

Triangle PQR is transformed to P′Q′R′ by matrix $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$.

Triangle P′Q′R′ is transformed to P″Q″R″ by matrix $\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$.

Find the single matrix that will transform PQR directly to P″Q″R″.

Solution

$$\begin{aligned} \begin{bmatrix} x'' \\ y'' \end{bmatrix} &= \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} 2 & 6 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \end{aligned}$$

The required matrix is $\begin{bmatrix} 2 & 6 \\ 1 & 5 \end{bmatrix}$.

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Further examples

EXAMPLE 5

Prove that $\begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}$ transforms **all** points in the x-y plane to the line $y = -2x$.

Solution

Consider some general point (a, b) transformed to (a', b').

$$\begin{bmatrix} a' \\ b' \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3a-b \\ -6a+2b \end{bmatrix}$$

$$= \begin{bmatrix} 3a-b \\ -2(3a-b) \end{bmatrix}$$

Thus for all image points: y-coordinate $= -2$ (x-coordinate).

Therefore all image points lie on the line $y = -2x$ as required.

EXAMPLE 6

All points on the line $y = 2x - 3$ are transformed by the matrix $\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$. Find the equation of the image line.

Solution

Consider some general point $(k, 2k - 3)$ on $y = 2x - 3$.

$$\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} k \\ 2k-3 \end{bmatrix} = \begin{bmatrix} 5k-6 \\ k \end{bmatrix}$$

Thus points (x, y) on the image line are of the form $x = 5k - 6$ and $y = k$.

Eliminating k gives $x = 5y - 6$, i.e. $y = 0.2x + 1.2$, the equation of the image line.

(Alternatively we could take any two points lying on $y = 2x - 3$, determine their images and find the equation of the straight line through these two images.)

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Exercise 11B

1 Matrix A represents a rotation of $-90°$ (i.e. $90°$ clockwise) about the origin, matrix B represents a rotation of $180°$ about the origin and matrix C represents a rotation of $90°$ (i.e. $90°$ anticlockwise) about the origin.

a Determine A, B and C.

Show that:

b $A^2 = B$ **c** $C^2 = B$ **d** $A^3 = C$

e $B^2 = I$ where I is the 2×2 identity matrix

f $A^{-1} = C$ **g** $B^{-1} = B$

2 Determine the 2×2 transformation matrix representing:

a a reflection in the x-axis,

b a reflection in the y-axis,

c a $180°$ rotation about the origin.

Hence show that a reflection in the x-axis followed by a reflection in the y-axis is the same as:

d a reflection in the y-axis followed by a reflection in the x-axis,

e a $180°$ rotation about the origin.

3 Matrix P represents a reflection in the line $y = -x$.

Determine P and show that matrix P is its own inverse.

4 A stretch parallel to the x-axis, scale factor 3, should multiply the area of an original shape by 3. Write down the 2×2 matrix representing this transformation and confirm that the determinant is of magnitude 3.

5 **a** Determine the matrix product $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 3 & 2 \\ 1 & 0 & 1 & -1 \end{bmatrix}$.

b Hence write the coordinates of A′, B′, C′ and D′ the images of A(0, 1), B(1, 0), C(3, 1) and D(2, –1) under the transformation $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$.

6 The triangle ABC is transformed to triangle A′B′C′ by the transformation matrix T where

$$T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

If A′, B′ and C′ have coordinates (7, 3), (3, 1) and (–2, –3) respectively, find the coordinates of A, B and C.

7 The triangle ABC is transformed to triangle A′B′C′ by the transformation matrix T where

$$T = \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix}.$$

If A′, B′ and C′ have coordinates (2, 0), (–2, 5) and (0, 2) respectively, find the coordinates of A, B and C.

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8 Triangle PQR is transformed to triangle P′Q′R′ by matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Triangle P′Q′R′ is transformed to triangle P″Q″R″ by matrix $\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$.

Find the single matrix that will transform PQR directly to P″Q″R″.

9 Triangle PQR is transformed to triangle P′Q′R′ by matrix $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$.

Triangle P′Q′R′ is transformed to triangle P″Q″R″ by matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

a Find the single matrix that will transform PQR directly to P″Q″R″.
b Find the single matrix that will transform P″Q″R″ directly to PQR.

10 Find the single 2×2 matrix representing the combination of a shear parallel to the y-axis, scale factor 3, followed by a clockwise rotation of 90° about the origin.

11 Find the single matrix representing the combination of a clockwise rotation of 90° about the origin, followed by a shear parallel to the y-axis, scale factor 3.

12 If the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ maps (1, 2) to (12, 7) and (–3, 1) to (–1, 0) then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 12 & -1 \\ 7 & 0 \end{bmatrix}$$

Determine a, b, c and d.

13 Quadrilateral ABCD is transformed to $A_1B_1C_1D_1$ by the matrix $\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$.

Quadrilateral $A_1B_1C_1D_1$ is transformed to $A_2B_2C_2D_2$ by the matrix $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$.

Quadrilateral $A_2B_2C_2D_2$ is transformed to $A_3B_3C_3D_3$ by the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

a Find the single matrix that will transform ABCD to $A_2B_2C_2D_2$.
b Find the single matrix that will transform ABCD to $A_3B_3C_3D_3$.
c Find the single matrix that will transform $A_2B_2C_2D_2$ to $A_1B_1C_1D_1$.
d Find the single matrix that will transform $A_3B_3C_3D_3$ to $A_1B_1C_1D_1$.

14 Use transformation matrices to prove that if a shape is
- reflected in the x-axis, and then
- reflected in the line $y = x$, and then
- rotated 90° clockwise about the origin,

it ends up in its original position.

15 The transformation matrix $T = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$ transforms the rectangle OABC to the parallelogram O′A′B′C′.

If O, A, B and C have coordinates (0, 0), (3, 0), (3, 2) and (0, 2) respectively, the area of OABC is 6 square units.

a Use the determinant of T to determine the area of O′A′B′C′.

b Determine the coordinates of O′, A′, B′ and C′.

c Draw OABC and O′A′B′C′ on square grid paper.

d Hence confirm your answer to part **a**, the area of O′A′B′C′.

16 The transformation matrix $M = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ transforms the square ABCD to the parallelogram A′B′C′D′.

A, B, C and D have coordinates (–2, 0), (0, –2), (2, 0) and (0, 2) respectively.

a With x and y-axes each from –6 to 6 show ABCD on grid paper.

b Determine the area of ABCD.

c Use the determinant of M to determine the area of A′B′C′D′.

d Show A′B′C′D′ on the grid and confirm that its area agrees with your answer from **b**.

17 Prove that the matrix $\begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}$ transforms all points on the line $y = 2x + 3$ to the single point (–3, 3).

18 Determine the equation of the image line formed when all points on the line $y = x - 1$ are transformed by the matrix $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$.

19 Prove that the matrix $\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$ transforms **all** points to the line $y = 3x$.

20 Show that the matrix $\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}$ transforms all points

a on the line $y = 5 - 3x$ to a single point, and find the coordinates of that point,

b in the x-y plane to one line, and find the equation of that line.

21 Prove that the transformation matrix $A = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}$ transforms the straight line $y = m_1x + p$ to the straight line $y = m_2x + p$ and find m_2 in terms of m_1.

A pair of straight lines are perpendicular to each other both before and after transformation by matrix A. Find the gradients of the two lines before the transformation.

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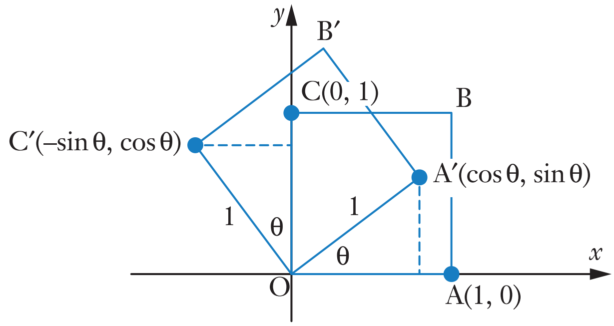
A general rotation about the origin

Rotations

The diagram below shows square OABC, of side one unit, rotated anticlockwise about the origin through an angle θ.



$(1, 0)$ maps to $(\cos\theta, \sin\theta)$ and $(0, 1)$ maps to $(-\sin\theta, \cos\theta)$

The required matrix is $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

A general reflection in a line that passes through the origin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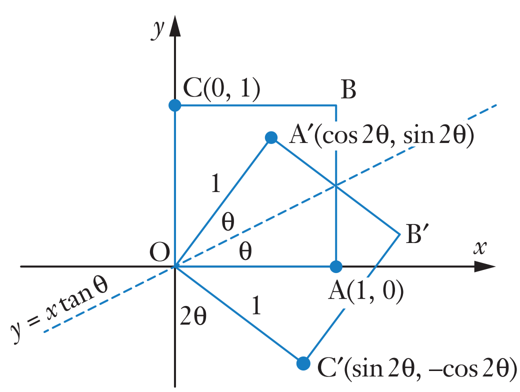
Reflections

Matrix multiplications

The diagram below shows square OABC, of side one unit, reflected in the straight line $y = mx$ where $m = \tan\theta$.



$(1, 0)$ maps to $(\cos 2\theta, \sin 2\theta)$ and $(0, 1)$ maps to $(\sin 2\theta, -\cos 2\theta)$

The required matrix is $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$.

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Exercise 11C

1 Find the 2×2 transformation matrix representing a rotation of

a 30° anticlockwise about the origin,

b 45° anticlockwise about the origin,

c 60° anticlockwise about the origin,

d 90° anticlockwise about the origin.

Use your answers to show that

e a 30° anticlockwise rotation about the origin followed by another 30° anticlockwise rotation about the origin is equivalent to a 60° anticlockwise rotation about the origin.

f a 30° anticlockwise rotation about the origin followed by a 60° anticlockwise rotation about the origin is equivalent to a 90° anticlockwise rotation about the origin.

g a 45° rotation about the origin followed by a 45° rotation about the origin is equivalent to a 90° rotation about the origin.

2 Find the 2×2 transformation matrix representing

a a reflection in the line $y = x\tan 30°$,

b a reflection in the line $y = x\tan 60°$.

Show that for each of parts **a** and **b** the square of the matrix is equal to the identity matrix and explain why this should be so.

3 Write a 2×2 matrix representing a clockwise rotation of θ about the origin.

4 Use transformation matrices to show that a reflection in the line $y = x \tan 45°$ followed by a reflection in the line $y = x\tan 60°$ is equivalent to an anticlockwise rotation about the origin of 30°.

5 By considering a rotation of angle A followed by a rotation of angle B, use the fact that the transformation matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

represents an anticlockwise rotation of θ about the origin to prove that

$$\sin(A+B) = \sin A \cos B + \cos A \sin B,$$

and

$$\cos(A+B) = \cos A \cos B - \sin A \sin B.$$

6 Prove that a reflection in the line $y = m_1x$, with $m_1 = \tan\theta$, followed by a reflection in the line $y = m_2x$, with $m_2 = \tan\phi$, is equivalent to an anticlockwise rotation about the origin of angle α, and find α in terms of θ and ϕ.

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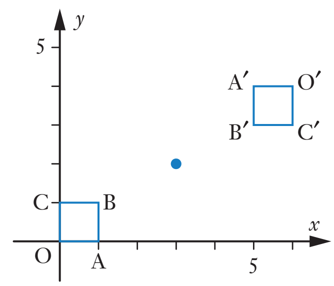
7 **a** The diagram on the right shows the unit square OABC and its image O′A′B′C′ after rotation of 180° about the point (3, 2).

By considering the transformation as a rotation about the origin followed by a translation, write the 180° rotation about (3, 2) in the form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} ? \\ ? \end{bmatrix}$$



b What 2 × 2 matrix will rotate O′A′B′C′ clockwise about the origin such that point O′ lies on the x-axis?

c Find the coordinates of O″, A″, B″ and C″, the images of O′, A′, B′ and C′ under the transformation described in part **b**.

Miscellaneous exercise eleven

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters in this unit, and the ideas mentioned in the Preliminary work section at the beginning of this unit.

1 Prove that $\cos^4\theta - \sin^4\theta = \cos 2\theta$.

2 Solve $2\cos^2 x + \sin x = 2\cos 2x$ for $0 \le x \le 2\pi$.

3 If $\cos 3\theta = a\cos^3\theta + b\cos^2\theta + c\cos\theta + d$ determine a, b, c and d.

4 When a shape is transformed under a reflection, or a rotation or a shear parallel to a coordinate axis, the area of the shape does not alter. Write the matrix representing each of the following transformations and confirm that for each matrix the absolute value of the determinant is equal to 1.

a Rotate 90° anticlockwise about the origin.

b Rotate 180° about the origin.

c Reflect in the x-axis.

d Reflect in the line $y = x$.

e Shear parallel to the x-axis, scale factor 4.

f Shear parallel to the y-axis, scale factor 3.

5 The matrices A, B and C shown below can be multiplied together to form a single matrix if A, B and C are placed in an appropriate order. What is the order and what is the single matrix this order produces?

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 3 \\ 3 & 1 & 4 & 0 \end{bmatrix}.$$

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6 If $A = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$, determine each of the following.

If any cannot be determined state this clearly and explain why.

a $A + B$ **b** $B + C$ **c** AC **d** CA

e BC **f** B^2 **g** $BA + C$

7 Explain why $\begin{bmatrix} 2x & -1 \\ 4 & x \end{bmatrix}$ cannot be a singular matrix for real x.

8 If $A = \begin{bmatrix} k & 4 \\ -3 & -1 \end{bmatrix}$, $A^2 + A = \begin{bmatrix} 0 & p \\ q & -12 \end{bmatrix}$ and $p > 0$, find k, p and q.

9 If $A = \begin{bmatrix} 1 & -2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ determine **a** AB **b** BA

10 Find x and y given that $\begin{bmatrix} 45 & y^2 \\ y^2 & 6x \end{bmatrix} - \begin{bmatrix} x^2 & y \\ -5y & 5 \end{bmatrix} = \begin{bmatrix} 4x & 4-y \\ -6 & x^2 \end{bmatrix}$.

11 With $A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$, what restrictions must be put on the values that x, y and z can take if we require $AB = BA$?

12 If $M = \begin{bmatrix} 0 & -1 \\ 2 & a \end{bmatrix}$ and $M^{-1}M^{-1} = \begin{bmatrix} b & 1 \\ c & d \end{bmatrix}$, find a, b, c and d.

13 **a** Determine the 2×2 transformation matrix that will transform the point (1, 0) to (3, –1) and the point (0, 1) to (–2, 1).

b Determine the 2×2 transformation matrix that will transform the point (2, 1) to (5, 3) and the point (1, –1) to (4, 0).

14 Triangle PQR is transformed to P′Q′R′ by matrix $\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$.

Triangle PQR is transformed to P″Q″R″ by matrix $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

Find the single matrix that will transform P′Q′R′ to P″Q″R″.

15 Find all solutions to the equation

$$\tan [2(x - 1.5)] = 2.3,$$

rounding answers to two decimal places when rounding is appropriate.

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